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Modular Conformal Scaling Group Evidenced in Lepton-Quark Mass

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Lepton-quark mass may reflect a correspondence in spacetime structure described by a modular conformal scaling group. Stemming in part from a spacetime line element correspondence $ds \rightarrow (\exp \lambda_n)\, ds$ in which the eight quantities $\lambda_0, \lambda_1, \ldots, \lambda_7$ constitute a closed set under a modular addition, the associated formula for lepton-quark mass (yielding values at the 1 GeV scale for the leptons and lighter quarks and at the physical pole for the top) is conjectured to be $m = m_f Q^2(\exp -\lambda_n)$, where $m_f = 10.245$ TeV is the progenitor fermion mass, Q is the charge number of the lepton or quark, and the modular group parameter λ_n is indexed by a fermion principal quantum number n that depends on three mutually independent projection operators.

1. INTRODUCTION

If it is not a low-energy approximation to a closed and complete quantum field theory, then the minimal three-generation standard model (SM) must be essentially somewhat phenomenological. Rather than being generated by field interaction and thus intrinsic to the Lagrangian, lepton-quark mass may be more primary and necessarily required to be put in by hand for SM. That lepton-quark mass may indeed be patently primary to SM is suggested by its variation from m_e for the electron to m_t ($>3.1 \times 10^5\, m_e$) for the top and its zero or nearzero value for the three neutrinos. As a consequence of this mass hierarchy, practical SM extensions encounter formidable technical difficulties which either encumber or preclude zero or nearzero neutrino mass, as required by experiment and theory (Dolgov and Rothstein, 1993). This so-called neutrino mass problem "vanishes" quite literally if fundamental fermion mass is actually modulated by the square of the charge number, m

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$\propto Q^2$. Interestingly enough, the proportionality of fermion mass to charge squared is a venerable notion, with perhaps the earliest antecedents in the classical $m_e = e^2/r_e$ electron self-energy constructs of Thomson and others (e.g., Born, 1957) more than 100 years ago. In contemporary SM the proportionality $m \propto Q^2$ is not derivable from the Lagrangian, although suggested by the quadratic form of gauge-invariant free-field electroweak energy and supported by the empirical relations $m_d/m_\mu \cong (1/3)^2$, $m_s/m_\tau \cong (1/3)^2$, as well as by $m_{\nu_e} \cong m_{\nu_\mu} \cong m_{\nu_\tau} << m_e$.

It has been observed that a simple mass formula of the form $m = \overline{m}\, Q^2(\exp \lambda)$ is wholly consistent with experimental measurements and quark model estimates for all 12 fundamental fermions (Rosen, 1995). Here $\overline{m}$ = 433.3 MeV is an input (mean fermion mass) constant, Q is the charge number of the lepton or quark, and λ is a real root of a quartic equation that brings in a principal quantum number n (= 0, 1, 2, 3). The physical basis for this mass formula and its relation to a discrete conformal correspondence in spacetime structure for lepton and quark states are analyzed in the present paper.

2. LEPTON-QUARK MASS AND MODULAR CONFORMAL SCALING

Consider the lepton-quark mass formula

$$m = m_f Q^2(\exp -\lambda_n) \tag{1}$$

as a conjectured result to be obtained from a preonlike or other structural model and to be employed as practical input in SM [i.e., mass values at the 1 GeV scale for the leptons and lighter quarks and at the physical pole for the top as in, e.g., Albright and Nandi (1994)]. In equation (1) m_f is the progenitor fermion mass, Q^2 (= 1, 0 for leptons, 4/9, 1/9 for quarks) is the *Thomson factor*, and λ_n is a modular group parameter that depends on a fermion principal quantum number n. The residue of a local conformal spacetime transformation $ds \rightarrow (\exp \lambda_n)\, ds$ which rescales the spacetime line element and thereby maps the progenitor fermion structural state into the lepton-quark structural states, the quantity $(\exp -\lambda_n)$ in (1) is an element of a finite (order 8 as shown below) discrete modular group (e.g., Biggs, 1985); the modular addition on the quantities $\lambda_0, \lambda_1, \ldots, \lambda_7$ is such that closure obtains: $\lambda_{n'} \oplus \lambda_{n''} = \lambda_n$ for all n', n'', and $n = f(n', n'') \equiv f(n'', n')$ contained in the set 0, 1, . . . , 7.

To formalize a suitable modular addition on the group parameter set λ_0, $\lambda_1, \ldots, \lambda_7$ it is necessary to express n in terms of three mutually commuting and simultaneously diagonalized projection operators p_0, p_1, p_2. With p_k =

$(p_k)^2$ for $k = 0, 1, 2$, the projection operator eigenvalues are 0 or 1, and $n = 0, 1, \ldots, 7$ according to the resolution identity

$$n \equiv \sum_{k=0}^{2} 2^k p_k \equiv p_0 + 2p_1 + 4p_2 \tag{2}$$

Hence n is simply given by $(p_2\, p_1\, p_0)_{\text{binary}}$ in the base-2 number system. The principal quantum number–projection operator association expressed by (2) is shown in Table I, and the modular addition on the λ_n then follows directly: when two p_k with the same k are added by $\oplus$, the eigenvalue of their modular sum is defined as the sum of their eigenvalues modulo 2,

$$p_k' \oplus p_k'' \equiv p_k = \begin{cases} 0 & \text{for} \quad p_k' = p_k'' \\ 1 & \text{for} \quad p_k' \neq p_k'' \end{cases} \tag{3}$$

Thus, with λ_n prescribed as a fixed linear combination of the projection operators

$$\lambda_n = c_0 p_0 + c_1 p_1 + c_2 p_2 \tag{4}$$

in which the c_k are constant (c-numbers), modular addition of the λ_n is closed by virtue of (3):

$$\begin{aligned} \lambda_{n'} \oplus \lambda_{n''} &= (c_0 p_0' + c_1 p_1' + c_2 p_2') \oplus (c_0 p_0'' + c_1 p_1'' + c_2 p_2'') \\ &\equiv c_0 p_0 + c_1 p_1 + c_2 p_2 = \lambda_n \end{aligned} \tag{5}$$

Table II shows the $\{\lambda_n\}$ modular addition explicitly for generic c_k; the associated modular group $\{(\exp -\lambda_n)\}$ is isomorphic to $C_2 \times C_2 \times C_2$, where C_2 is the two-group.

Admissible $|Q|$ values for the fermions are given by the projection operator eigenvalues (p_0, p_1, p_2) via the formula

$$|Q| = \begin{cases} (p_{(1-p_2)} + 2p_2)/3 \\ \text{or} \\ (p_{(1-p_2)} + 2p_{p_2})/3 \end{cases} \tag{6}$$

or equivalently by

$$|Q| = \begin{Bmatrix} p_1/3 \\ \text{or} \\ (p_1 + 2p_0)/3 \end{Bmatrix} \quad \text{for} \quad p_2 = 0$$

$$|Q| = \begin{Bmatrix} (p_0 + 2)/3 \\ \text{or} \\ (p_0 + 2p_1)/3 \end{Bmatrix} \quad \text{for} \quad p_2 = 1 \tag{7}$$

Table I. Lepton-Quark Projection Operator Eigenvalues (p_0, p_1, p_2), Associated Principal Fermion Quantum Numbers n [by (2)], Admissible Charge Magnitudes $|Q|$ [by (6)], Group Parameter Eigenvalues λ_n [by (11)], and Mass Values According to (1) with $m_f = 10.2452$ TeV as Prescribed Input

(p_0, p_1, p_2)	(0, 0, 0)	(1, 0, 0)	(0, 1, 0)	(1, 1, 0)	(0, 0, 1)	(1, 0, 1)	(0, 1, 1)	(1, 1, 1)
n	0	1	2	3	4	5	6	7
$\|Q\|$	0	$0, \frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}, 1$	$0, \frac{2}{3}$	$\frac{1}{3}, 1$	$\frac{2}{3}$	1
λ_n	0	$\frac{1}{2} + 2\sqrt{2}$	$\frac{5}{2} + 2\sqrt{2}$	$3 + 4\sqrt{2}$	$\frac{5}{2} + 4\sqrt{2}$	$3 + 6\sqrt{2}$	$5 + 6\sqrt{2}$	$\frac{11}{2} + 8\sqrt{2}$
m (MeV)	0	0, 163,238	5523.0	197.99, 1781.96	0, 1305.8	11.703, 105.324	6.3351	0.51100
Fermion symbol	ν_τ	ν_μ, t	b	s, τ	ν_e, c	d, μ	u	e

Table II. Modular Addition for λ_n's Implied by (5) and (2) with c_k's Generic[a]

$\oplus$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
λ_1	λ_1	λ_0	λ_3	λ_2	λ_5	λ_4	λ_7	λ_6
λ_2	λ_2	λ_3	λ_0	λ_1	λ_6	λ_7	λ_4	λ_5
λ_3	λ_3	λ_2	λ_1	λ_0	λ_7	λ_6	λ_5	λ_4
λ_4	λ_4	λ_5	λ_6	λ_7	λ_0	λ_1	λ_2	λ_3
λ_5	λ_5	λ_4	λ_7	λ_6	λ_1	λ_0	λ_3	λ_2
λ_6	λ_6	λ_7	λ_4	λ_5	λ_2	λ_3	λ_0	λ_1
λ_7	λ_7	λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0

[a]Associated group is $C_2 \times C_2 \times C_2$ with C_2 the two-group

The $|Q|$ values prescribed by (6) or (7) are shown for their respective (p_0, p_1, p_2) in Table I.

Reflecting the scaling strengths of the p_k in the physical conformal correspondence that relates fermion structural states, the three c_k in (4) should be simply related numbers in the base-2 system, like the coefficients 2^k in (2), if equation (1) is a fundamental scaling law. Indeed by empirical inspection one finds that the c_k appear to satisfy the simple base-2 linear conditions

$$c_k - c_{k-1} = 2^{(k+1)/2} \qquad (k = 1, 2) \tag{8}$$

$$c_0 + c_1 - c_2 = 1/2$$

Conversely, the c_k are fixed by (8) as

$$c_0 = \frac{1}{2} + 2\sqrt{2}, \qquad c_1 = \frac{5}{2} + 2\sqrt{2}, \qquad c_2 = \frac{5}{2} + 4\sqrt{2} \tag{9}$$

and hence (4) is expressed explicitly as

$$\lambda_n = \left(\frac{1}{2} + 2\sqrt{2}\right)p_0 + \left(\frac{5}{2} + 2\sqrt{2}\right)p_1 + \left(\frac{5}{2} + 4\sqrt{2}\right)p_2 \tag{10}$$

From (10) and (2) one finds that

$$\lambda_0 = 0 \qquad \lambda_4 = \frac{5}{2} + 4\sqrt{2}$$

$$\lambda_1 = \frac{1}{2} + 2\sqrt{2} \qquad \lambda_5 = 3 + 6\sqrt{2}$$

$$\lambda_2 = \frac{5}{2} + 2\sqrt{2} \qquad \lambda_6 = 5 + 6\sqrt{2}$$

$$\lambda_3 = 3 + 4\sqrt{2} \qquad \lambda_7 = \frac{11}{2} + 8\sqrt{2} \tag{11}$$

By substituting the latter λ_n values into formula (1) and prescribing m_f = 10.2452 TeV as input (in order to have m_e = 0.51100 MeV), one obtains the mass values shown in Table I.

All of the theoretical masses given by (1) and (11) and displayed in Table I are in satisfactory agreement with direct experiments and quark model estimates (Aguilar-Benitez *et al.*, 1992; Bai *et al.*, 1992; Dominguez and de Rafael, 1987; Gasser and Leutwyler, 1982; Donoghue and Holstein, 1992). In particular, the muon mass m_μ = 105.324 MeV is 0.316% below the experimental value, while the tau mass m_τ = 1781.96 MeV is between the BES experimental value (Bai *et al.*, 1992) of 1776.9 MeV and the previous 8-year world mean of 1784.1 MeV. The theoretical top mass m_t = 163.238 GeV is in the lower part of the range delineated recently by the CDF (1994). In fact the masses given by (1) and (11) may be accurate to better than 0.159% for all charged leptons[2] and quarks if all quark mass values except for the top (physical pole mass) are understood to pertain to a certain value slightly below the 1 GeV scale. The special (physical pole mass) scale for the top that appears in this (renormalization-independent) conformal scaling group model may be related to the hierarchal (n = 1) positioning of the top.

3. SUMMARY

Formula (1), which gives experimentally admissible zero mass for the three neutrinos and accurately consistent mass values for the charged leptons and quarks over the five-order-of-magnitude range characterized by the ratio $m_t/m_e \cong 3.2 \times 10^5$, supports the existence of a discrete conformal correspondence between fermion structural states, as well as the basic Thomson factor Q^2 in lepton-quark mass. The modular group parameter λ_n is indexed by the fermion principal quantum number[3] and expressed by (4) in terms of the mutually independent projection operators (p_0, p_1, p_2) and the c_k. To complete this *aufbauprinzip* (Wigner, 1959) for generating lepton-quark mass from the Thomson-modulated primordial fermion mass, the c_k are fixed by the simple base-2 linear conditions in (8), conditions which may relate to the details of the structural theory for leptons and quarks. Finally it is noteworthy that the primordial fermion f with mass m_f = 10.245 TeV in (1) may be evidenced

[2] If m_f in (1) is set equal to 10.2614 TeV as prescribed input, then the theoretical masses for the electron and muon would both be within 0.159% of their respective experimental values.

[3] All other lepton and quark quantum numbers are simply expressible in terms of Q and n; in particular, the baryon number B (= 0 for leptons, 1/3 for quarks, −1/3 for antiquarks) is independent of n and given by $B = 9(1 - |Q|)(|Q| - \frac{1}{2})Q$.

corroboratively if the primordial fermion pair annihilation $f + \bar{f} \rightarrow \gamma + \gamma$ can be detected in cosmic radiation.

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